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1994 J. Phys. A: Math. Gen. 27 L617

(http://iopscience.iop.org/0305-4470/27/17/004)

J. Phys. A: Math. Gen. 27 (1994) L617–L623. Printed in the UK

LETTER TO THE EDITOR

On the quantization of the constrained AKNS flows

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Received 14 April 1994

Abstract. It is shown that the quantum version of the constrained Ablowitz–Kaup–Newell–Sogur (AKNS) flows from a completely integrable system and the corresponding Lax pair can be constructed. The quantum inverse problem for the system is then formulated following the technique of Jurgo and Gaunoulis. Finally the algebraic Bethe ansatz is explicitly constructed, thereby determining the energy eigenvalue of the excited states.

Derivation of completely integrable dynamical systems by constraining the flows generated by Ablowitz–Kaup–Newell–Sogur (AKNS) or other types of Lax operators has attracted the attention of many researchers [1]. It has even been demonstrated that such dynamical systems are bi-Hamiltonian with two separate symplectic structures [2]. Complete integrability of such systems was ascertained through the derivation of an infinite number of conserved quantities and the corresponding Lax pair [3]. In this letter we show that it is possible to derive a quantum mechanical version of such constrained dynamics by using the commutation rules for canonical variable and a Lax operator. Incidentally, complete integrability of the quantized system can also be understood through the construction of infinite numbers of conserved quantities in involution. In the latter part of this letter we have formulated a quantum inverse problem for the situation by following the methodology of Jurco and Ganoulis and have constructed the Bethe ansatz.

The AKNS hierarchy is generated by the Lax pair;

$$\Psi_x = L\Psi \qquad \Psi_t = V\Psi \tag{1}$$

where

$$L = \begin{pmatrix} \lambda & r \\ s & -\lambda \end{pmatrix} \qquad \text{and} \qquad V = \sum_n \lambda^n V_n .$$

The first classical restricted flow leads to a dynamical system whose equation can be written as;

$$\begin{aligned} q_x &= \Delta q - 2(qQ)P = \partial H_1/\partial p \\ Q_x &= \Delta Q - 2(qQ)p = \partial H_1/\partial p \\ p_x &= -\Delta p + 2(qQ)Q = -\partial H_1/\partial q \\ P_x &= -\Delta P + 2(pQ)q = -\partial H_1/\partial Q \end{aligned} \tag{2}$$

where

$$H_1 = q\Delta P + Q\Delta P - 2(qQ)(pP) \tag{3}$$

and $q = (q_l \ldots q_n)$, $p = (p_l \ldots p_n)$, etc are n-component vectors with $\Delta = \mathrm{diag}(\xi_1 \ldots \xi_k)$. Also

$$(q\Delta p) = \sum_{k=1}^{n} q_k \xi_k p_k. \tag{4}$$

It has been shown in [4] that these equations have Lax pair (M, u), such that;

$$M_x = [u, M] \tag{5}$$

with

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \sum_{k=1}^{n} (\lambda - \xi_k)^{-1} \begin{bmatrix} q_k p_k + Q_k P_k & -2q_k Q_k \\ 2p_k P_k & -(q_k p_k + Q_k P_k) \end{bmatrix} \tag{6}$$

and

$$u = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}.$$

The complete integrability of equation (2) was analysed with the help of (6). From the form of (2) it is quite evident that $(p.q)$ and (P, Q) are canonical pairs of variables in the classical sense. Our first motivation is to consider the set of nonlinear dynamical equations *which can be obtained if we consider the set* $(p.q)$ *and* (P, Q) *as quantum mechanical* variables.

The quantum mechanical version of equations (2) can be obtained from the symmetrized form of the Hamiltonian H_1 written as

$$H_1^q = \sum_k (q_k \xi_k p_k + Q_k \xi_k P_k) - \sum_{m,n} \{(q_m Q_m)(p_n P_n) + (p_m P_m)(q_n Q_n)\} \tag{7}$$

supplemented with the commutation rules

$$[q_l, p_m] = i\hbar \delta_{lm}$$

$$[Q_l, P_m] = i\hbar \delta_{lm}.$$

In (7) we have not symmetrized the terms in the first bracket since it gives rise to a constant term only. The quantum mechanical equations for the dynamical system are;

$$q_{lx} = \frac{i}{\hbar}[H_1^q, q_l] \qquad Q_{lx} = \frac{i}{\hbar}[H_1^q, Q_l] \tag{8}$$

whence we get;

$$\begin{aligned}
q_{lx} &= q_l \xi_l - \sum_m (q_m Q_m) P_l - P_l \sum_m (q_m Q_m) \\
p_{lx} &= -\xi_l p_l + Q_l \sum_n (p_n P_n) + \sum_n (p_n P_n) Q_l \\
P_{lx} &= -\xi_l p_l + q_l \sum_n p_n P_n + \sum_n (p_n P_n) q_l \\
Q_{lx} &= Q_l \xi_l - \sum_m (q_m Q_m) p_l - p_l \sum_m q_m Q_m
\end{aligned} \tag{9}$$

where we have used the fundamental commutator;

$$[q_n, p_m] = i\hbar\delta_{nm}$$
$$[Q_n, P_m] = i\hbar\delta_{nm}. \tag{10}$$

The quantum mechanical system (9) can also be associated with a Lax equation which we write as;

$$M_x = [u, M] \tag{11}$$

where (10) is to be used for all calculations. For example, the '21' matrix coefficient of equation (11) gives

$$2\sum_{l=1}^{n}(\lambda - \xi_l)^{-1}(p_l P_l)_x = 4\sum_{l=1}^{n} p_l P_l - 4\lambda\sum_{l=1}^{n}(\lambda - \xi_l)^{-1}(p_l P_l)$$
$$+ 2\sum_{k,l=1}^{n}(\lambda - \xi_l)^{-1}\{(p_k P_l)(q_l p_l + Q_l P_l) + (q_l p_l + Q_l P_l)(p_k P_k)\}. \tag{12}$$

Using $\frac{\lambda}{\lambda-\xi_l} = 1 + \frac{\xi_l}{\lambda-\xi_l}$, upon simplification we get

$$(p_l P_l)_x = -2\xi_l p_l P_l + \sum_{k=1}^{n}\{(p_k P_k), (p_l P_l + Q_l P_l)\}_+ \tag{13}$$

where $(a, b)_+$ stands for $ab + ba$.

It can be easily seen that equation (13) is a simple consequence of (9). So we can say that equation (11) represents the Lax equation of the quantum mechanical flow given by (9). Now for brevity we define:

$$\sigma_k = q_k p_k + Q_k P_k \qquad \zeta_k = q_k Q_k \qquad \eta_k = p_k P_k.$$

We now observe that the conserved quantities are obtained as the coefficient of λ^{-m} in $\mathrm{Tr}[M^n]$. For example;

$$\mathrm{Tr}\, M^2 = 2 + 4\sum_k \frac{\sigma_k}{(\lambda - \xi_k)} + 2\sum_k \frac{\sigma_k^2 - 4\eta_k\zeta_k}{(\lambda - \xi_k)^2}$$
$$+ \sum_{\substack{k,r\\ k\neq r}}\left[\frac{1}{\lambda - \xi_k} - \frac{1}{\lambda - \xi_r}\right]\frac{1}{(\xi_k - \xi_r)}(2\sigma_k\sigma_r - 4\zeta_k\eta_r - 4\eta_k\zeta_r). \tag{14}$$

So coefficients of different powers of λ^{-n} yield;

coefficient of λ^{-1}

$$C_{-1} = 4\sum_k \sigma_k$$

coefficient of λ^{-2} (15)

$$C_{-2} = 4\sum_k \zeta_k\sigma_k + 2\sum_{kr}\sigma_k\sigma_r - 4\sum_{kr}(\eta_k\zeta_r + \zeta_k\eta_r)$$

and one can easily identify that the Hamiltonian H, written in equation (8) as;

$$H_1^q = \tfrac{1}{4}C_{-2} - \tfrac{1}{32}C_{-1}^2. \tag{16}$$

So we have, until now, constructed a quantum mechanical version of (2) and also a Lax pair for it, and have also shown that the system has many integrals of motion.

We can now formulate the quantum inverse problem with the help of the above Lax pair. We compute $[M^1(\lambda), M^2(\mu)]$ where $M^1(\lambda) = M(\lambda)\otimes 1$, $M^2(\mu) = 1\otimes M(\mu)$, whence we get;

$$[M^1(\lambda), M^2(\mu)] = \sum_{k,l=1}^{n} (\lambda - \xi_k)^{-1}(\lambda - \xi_l)^{-1}\delta_{kl}\cdot J \tag{17}$$

where J is a 4×4 matrix given by

$$J = \begin{bmatrix} 0 & 4i\hbar q_k Q_k & -4i\hbar q_k Q_k & 0 \\ 4i\hbar p_k P_k & 0 & -4i\hbar(q_k p_k + Q_k P_k) & 4i\hbar q_k Q_k \\ -4i\hbar p_k P_k & 4i\hbar(q_k p_k + Q_k P_k) & 0 & -4i\hbar q_k Q_k \\ 0 & 4i\hbar p_k P_k & -4i\hbar p_k P_k & 0 \end{bmatrix}$$

and the right-hand side of (17) is seen to be equal to

$$[r(\lambda-\mu), M^1(\lambda) + M^2(\mu)] \tag{18}$$

where $r(\lambda-\mu) = -\frac{i\hbar}{\lambda-\mu}P$. P being the permutation matrix. Note that we have derived the r-matrix using the commutation rules, not the Poisson brackets as such. We can now proceed to set-up the algebraic Bethe ansatz equation following Jurco [6] and Ganoulis [5]. In the compact notation $M(\lambda)$ can be written as;

$$\begin{aligned} M(\lambda) &= \begin{bmatrix} 1 + \sum_{k=1}^{n}(\lambda-\xi_k)^{-1}\sigma_k & -2\sum_{k=1}^{n}(\lambda-\xi_k)^{-1}\xi_k \\ 2\sum_{k=1}^{n}(\lambda-\xi_k)^{-1}n_k & -1 - \sum_{k=1}^{n}(\lambda-\xi_k)^{-1}\sigma_k \end{bmatrix} \\ &= \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \text{(say)} \end{aligned} \tag{19}$$

so that

$$\begin{aligned} T(\lambda) &= \tfrac{1}{2}\operatorname{Tr} M^2(\lambda) \\ &= \tfrac{1}{2}[A^2(\lambda) + D^2(\lambda) + B(\lambda)C(\lambda) + C(\lambda)B(\lambda)]. \end{aligned} \tag{20}$$

On the other hand, equation (18) leads to

$$\begin{aligned} [A(\lambda), B(\mu)] &= \frac{2i\hbar}{\lambda-\mu}(B(\mu) - B(\lambda)) \\ [D(\lambda), B(\mu)] &= \frac{2i\hbar}{\lambda-\mu}(B(\lambda) - B(\mu)) \\ [B(\lambda), C(\mu)] &= \frac{2i\hbar}{\lambda-\mu}(D(\lambda) - A(\lambda) + A(\mu) - D(\mu)). \end{aligned} \tag{21}$$

In the following we will set $A(\lambda) = -D(\lambda)$. Equation (21) yields at once

$$[T(\lambda), B(\mu)] = \frac{4i\hbar}{\lambda - \mu}[B(\mu)A(\lambda) - B(\lambda)A(\mu)]. \tag{22}$$

The construction of the Bethe states starts with the assumption of the existence of the pseudo-vacuum state viz $|0\rangle$ such that

$$C(\lambda)|0\rangle = 0 \qquad A(\lambda)|0\rangle = a(\lambda)|0\rangle \tag{23}$$

and

$$T(\lambda)|0\rangle = t(\lambda)|0\rangle.$$

Now, starting with (21), if we set $\mu = \lambda + \varepsilon$, ε being small, we get

$$B(\lambda)C(\lambda+\varepsilon) - C(\lambda+\varepsilon)B(\lambda) = -\frac{4i\hbar}{\varepsilon}[A(\lambda+\varepsilon) - A(\lambda)] \tag{24}$$

whence, in the limit $\varepsilon \to 0$, we get

$$C(\lambda)B(\lambda) = B(\lambda)C(\lambda) + 4i\hbar A'(\lambda) \tag{25}$$

where $A'(\lambda) = \partial A/\partial\lambda$.

So, using equation (20) and (23),

$$\begin{aligned} t(\lambda)|0\rangle &= \tfrac{1}{2}[2A^2(\lambda) + B(\lambda)C(\lambda) + C(\lambda)B(\lambda)]|0\rangle \\ &= [a^2(\lambda) + 2i\hbar a'(\lambda)]|0\rangle. \end{aligned} \tag{26}$$

Our motivation is to determine the energy eigenvalue of this state and the equations satisfied by the eigenmomenta $(\mu_1 \ldots \mu_n)$ and observe

$$\begin{aligned} T(\lambda)|\mu_1\rangle &= T(\lambda)B(\mu_1)|0\rangle \\ &= \left\{t(\lambda) + 4i\hbar\frac{a(\lambda)}{\lambda - \mu_1}\right\}|\mu_1\rangle - 4i\hbar\frac{a(\mu_1)}{\lambda - \mu_1}|\lambda\rangle \end{aligned} \tag{27}$$

so that the single-particle state has got energy

$$E_1 = t(\lambda) + 4i\hbar\frac{a(\lambda)}{\lambda - \mu_1} \tag{28}$$

whereas the corresponding momentum μ_1 is determined by

$$a(\mu_1) = 0 \tag{29}$$

which comes from the usual vanishing condition for the 'unwanted terms'. Proceeding now to two-particle states

$$B(\mu_1)B(\mu_2)|0\rangle = |\mu_1, \mu_2\rangle \tag{30}$$

we get, using the commutation rules,

$$T(\lambda)|\mu_1, \mu_2\rangle = T(\lambda)B(\mu_1)B(\mu_2)|0\rangle.$$

$$\begin{aligned} T(\lambda)|\mu_1, \mu_2\rangle = &\left\{t(\lambda) + 4i\hbar\left(\frac{a(\lambda)}{\lambda-\mu_1} + \frac{a(\lambda)}{\lambda-\mu_2}\right) - \frac{8\hbar^2}{(\lambda-\mu_1)(\lambda-\mu_1)}\right\}|\mu_1, \mu_2\rangle \\ &+ \left\{-4i\hbar\frac{a(\mu_2)}{\lambda-\mu_2} + 8\hbar^2\frac{1}{(\lambda-\mu_2)(\mu_1-\mu_2)}\right\}|\lambda, \mu_1\rangle \\ &+ \left\{-4i\hbar\frac{a(\mu_1)}{\lambda-\mu_1} + 8\hbar^2\frac{1}{(\lambda-\mu_1)(\mu_1-\mu_2)}\right\}|\lambda, \mu_2\rangle. \end{aligned} \quad (31)$$

So we immediately infer that the two particle state has got energy;

$$E_2 = t(\lambda) + 4i\hbar\left(\frac{a(\lambda)}{\lambda-\mu_1} + \frac{a(\lambda)}{\lambda-\mu_2}\right) - \frac{8\hbar^2}{(\lambda-\mu_1)(\lambda-\mu_2)} \quad (32)$$

along with the following two subsidiary conditions determining the eigenmomenta, μ_1, μ_2.

$$-4i\hbar\frac{a(\mu_2)}{\lambda-\mu_2} + 8\hbar^2\frac{1}{(\lambda-\mu_2)(\mu_2-\mu_1)} = 0 \quad (33)$$

and

$$4i\hbar\frac{a(\mu_1)}{\lambda-\mu_1} + 8\hbar^2\frac{1}{(\lambda-\mu_1)(\mu_1-\mu_2)} = 0.$$

A *similar computation can be easily carried out for the n*-particle Bethe state and the energy of such a state is given as;

$$E_n = t(\lambda) + 4i\hbar\sum_{i=1}^{n}\frac{a(\lambda)}{\lambda-\mu_1} - 8\hbar^2\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{(\lambda-\mu_i)(\lambda-\mu_j)} \quad (34)$$

where the eigenmomenta $\mu_1 \ldots \mu_n$ are determined by n equations of the form (33), which when simplified read;

$$a(\mu_\alpha) = -2ih\sum_{\substack{i=1\\ i\neq\alpha}}^{n}\frac{1}{\mu_\alpha-\mu_i} \qquad (\alpha = 1, \ldots, n). \quad (35)$$

In the above expression $t(\lambda)$ *stands for the quantity* $a^2(\lambda) + 2i\hbar a'(\lambda)$.

So in the above analysis we have shown that it is possible to set up a quantum mechanical version of the constrained dynamical system determined by the AKNS flow. It is interesting to note that a Lax pair which has the same form as its classical counterpart can reproduce the quantum mechanical system, with the difference that the entries of Lax matrices are no longer commutative quantities. Such an L operator is then used to generate a r-matrix, not through Poisson bracket but by commutation rules which in turn has been used to set up an algebraic Bethe ansatz, determining the n-particle excited state. It will be really interesting to see whether similar considerations hold good for other constrained dynamics studied by several authors.

One of the authors (AGC) is grateful to CSIR (Government of India) for a JRF.

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